108.3.27修訂

國立虎尾科技大學
106-2期中會考解答

(一)填充題:

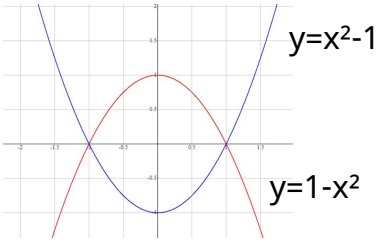
1. 求$\int \cos^{10} x \sin^3 x \, dx =$

解:

$$\int \cos^{10} x \sin^3 x \, dx = \int \cos^{10} x \sin^2 x \cdot \sin x \, dx = -\int \cos^{10} x(1-\cos^2 x)\, d\cos x$$
$$= -\int (\cos^{10} x - \cos^{12} x)\, d\cos x = -\frac{1}{11}\cos^{11} x + \frac{1}{13}\cos^{13} x + c$$

2. 求由$y = 1 - x^2$與$y = x^2 - 1$所圍封閉區域面積=

解:



先求交點, 即聯立方程式

$$\begin{cases} y = 1 - x^2 \\ y = x^2 - 1 \end{cases} \Longrightarrow (-1,0),(1,0) \text{ 為交點}$$

長度為$l(x) = (1-x^2) - (x^2-1) = -2x^2 + 2$

$$A = \int_{-1}^{1} (-2x^2 + 2)dx = -\frac{2}{3}x^3 + 2x\Big|_{-1}^{1} = \frac{8}{3}$$

3. 求$\int \sqrt[3]{\tan x} \sec^4 x \, dx =$

解:

$$\int \sqrt[3]{\tan x} \sec^4 x \, dx = \int \tan^{\frac{1}{3}} x(\tan^2 x + 1)\sec^2 x \, dx = \int \tan^{\frac{7}{3}} x + \tan^{\frac{1}{3}} x \, d\tan x$$
$$= \frac{3}{10}\tan^{\frac{10}{3}} x + \frac{3}{4}\tan^{\frac{4}{3}} x + c$$

4. 求$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx =$

解:

令$u = \sqrt{x} \Longrightarrow du = \frac{1}{2\sqrt{x}}dx \Longrightarrow 2du = \frac{1}{\sqrt{x}}dx$

$$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int_1^2 e^u \cdot 2du = 2e^u\Big|_1^2 = 2(e^2 - e)$$

5. 求$\int \frac{\sqrt{x^2-9}}{x^2}\,dx =$

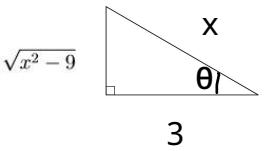
解:

令 $x = 3\sec\theta \Longrightarrow dx = 3\sec\theta\tan\theta d\theta$ ，且 $\sqrt{x^2-9} = 3\tan\theta$

$$\int \frac{\sqrt{x^2-9}}{x^2}\,dx = \int \frac{3\tan\theta}{9\sec^2\theta}\cdot 3\sec\theta\tan\theta d\theta = \int \frac{\tan^2\theta}{\sec\theta}d\theta = \int \frac{\sec^2\theta-1}{\sec\theta}d\theta$$

$$= \int (\frac{\sec^2\theta}{\sec\theta} - \frac{1}{\sec\theta})d\theta = \int (\sec\theta - \cos\theta)d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + c$$

$$= \ln|\frac{x+\sqrt{x^2-9}}{3}| - \frac{\sqrt{x^2-9}}{x} + c$$



6. 求$\int \frac{\sin(\ln x)}{x}\,dx =$

解:

令 $u = \ln x \Longrightarrow du = \frac{1}{x}dx \Longrightarrow xdu = dx$

$$\int \frac{\sin(\ln x)}{x}\,dx = \int \frac{\sin u}{x}x\,du = \int \sin u\,du = -\cos u + c = -\cos(\ln x) + c$$

7. 求$\int \frac{2x+3}{2x^2-x}\,dx =$

解:

令 $\frac{2x+3}{x(2x-1)} = \frac{A}{x} + \frac{B}{2x-1}$

$A = \frac{2x+3}{2x-1}\Big|_{x=0} = -3$ ， $B = \frac{2x+3}{x}\Big|_{x=\frac{1}{2}} = 8$

$$\int \frac{2x+3}{2x^2-x}\,dx = \int (\frac{-3}{x} + \frac{8}{2x-1})\,dx = -3\ln|x| + 4\ln|2x-1| + c$$

8. 求$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx =$

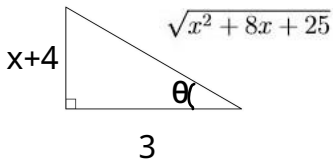
解:

$\because x^2+8x+25 = (x+4)^2+3^2$ $\therefore$ 令 $x+4 = 3\tan\theta \Longrightarrow dx = 3\sec^2\theta d\theta$

且 $x^2+8x+25 = 9\sec^2\theta$

$$\int \frac{1}{\sqrt{x^2+8x+25}}\,dx = \int \frac{1}{\sqrt{9\sec^2\theta}}\cdot 3\sec^2\theta = \int \sec\theta d\theta$$

$$= \ln|\sec\theta+\tan\theta| + c = \ln\left|\frac{x+4+\sqrt{x^2+8x+25}}{3}\right| + c$$



9. 求$\int \tan^{-1} x\,dx =$
解:

$$\begin{array}{lll} f(x)=\tan^{-1}x & + & g'(x)=1 \\ & \searrow & \\ f'(x)=\dfrac{1}{1+x^2} & - & g(x)=x \\ & \longleftarrow & \end{array}$$

$$\int \tan^{-1} x\,dx = x\tan^{-1} - \int \frac{x}{1+x^2}\,dx \ \ (\text{令 } u=1+x^2 \Longrightarrow du=2xdx \Longrightarrow dx=\frac{1}{2x}du)$$
$$= x\tan^{-1} - \int \frac{x}{u}\cdot\frac{1}{2x}du = x\tan^{-1}x - \int \frac{1}{2u}du = x\tan^{-1}x - \frac{1}{2}\ln|1+x^2| + c$$

10. 求$\int \sqrt{x}\ln x\,dx =$
解:

$$\begin{array}{lll} f(x)=\ln x & + & g'(x)=\sqrt{x} \\ & \searrow & \\ f'(x)=\dfrac{1}{x} & - & g(x)=\frac{2}{3}x^{\frac{3}{2}} \\ & \longleftarrow & \end{array}$$

$$\int \sqrt{x}\ln x\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \int \frac{2}{3}x^{\frac{3}{2}}\cdot\frac{1}{x}\,dx = \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}dx$$
$$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c$$

11. 求$\int \dfrac{\tan^{-1}x}{x^2+1}\,dx =$
解:

令 $u=\tan^{-1}x \Longrightarrow du=\dfrac{1}{1+x^2}dx \Longrightarrow dx=(1+x^2)du$

$$\int \frac{\tan^{-1}x}{x^2+1}\,dx = \int \frac{u}{x^2+1}(1+x^2)\,du = \int u du = \frac{1}{2}u^2 + c = \frac{1}{2}(\tan^{-1}x)^2 + c$$

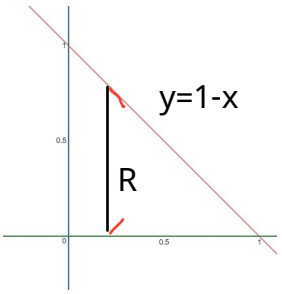
12. 求$\int (2t^{-2}-3t^{-1})\,dt =$
解:

$$\int (2t^{-2}-3t^{-1})\,dt = -2t^{-1} - 3\ln|t| + c = -\frac{2}{t} - 3\ln|t| + c$$

13. 求$\int(4\cos x+2\sin x-\frac{3}{x})\,dx=$
解:

$$\int(4\cos x+2\sin x-\frac{3}{x})\,dx=4\sin x-2\cos x-3\ln|x|$$

14. 求由$y=1-x,x=0,y=0$所圍封閉區域繞x軸旋轉之體積=
解:



半徑$R=y=1-x$
面積$A(x)=\pi(1-x)^2=\pi(x^2-2x+1)$
體積$V=\pi\int_0^1(x^2-2x+1)\,dx=\pi(\frac{1}{3}x^3-x^2+x\Big|_0^1)=\frac{\pi}{3}$

(二)計算題:

1. 試回答下列問題
(1) 利用半角公式將 $\sin^4 x$寫成恆等式 $\sin^4 x=a\cos 4x+b\cos 2x+d$,
求常數 a,b及d之值為何? (3分)
(2) 求不定積分$\int\sin^4 x\,dx$=? (7分)
解:

(1) $\sin^4 x=(\frac{1-\cos 2x}{2})^2=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}\cos^2 2x$
$=\frac{1}{4}-\frac{1}{2}\cos 2x+\frac{1}{4}(\frac{1+\cos 4x}{2})=\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x$
$\therefore a=\frac{1}{8}\ ,\ b=-\frac{1}{2}\ ,\ d=\frac{3}{8}$

(2) $\int\sin^4 x dx=\int(\frac{3}{8}-\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x)dx=\frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+c$

2. 已知$f(x)=\dfrac{x^3-2x^2+3x+1}{(x-1)^4}$, (1)將$f(x)$化為部分分式.(4分)
(2)利用結果(1)求$\int f(x)dx$(6分)

解:

(1) 令 $\dfrac{x^3-2x^2+3x+1}{(x-1)^4}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{(x-1)^3}+\dfrac{D}{(x-1)^4}$

$g(x)=x^3-2x^2+3x+1=A(x-1)^3+B(x-1)^2+C(x-1)+D$

$g(1)=D=3$

$g'(x)=3x^2-4x+3=3A(x-1)^2+2B(x-1)+C$

$g'(1)\Longrightarrow C=2$

$g''(x)=6x-4=6A(x-1)+2B$

$g''(1)\Longrightarrow B=1$

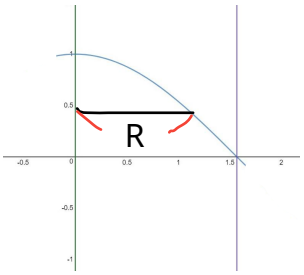
$g'''(x)=6=6A\Longrightarrow A=1$

$\therefore \dfrac{x^3-2x^2+3x+1}{(x-1)^4}=\dfrac{1}{x-1}+\dfrac{1}{(x-1)^2}+\dfrac{2}{(x-1)^3}+\dfrac{3}{(x-1)^4}$

(2) $\displaystyle\int f(x)dx=\int\left[\frac{1}{x-1}+\frac{1}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{3}{(x-1)^4}\right]dx$

$=\ln|x-1|-\dfrac{1}{x-1}-\dfrac{1}{(x-1)^2}-\dfrac{1}{(x-1)^3}+c$

3. 求由$y=\cos x, 0\le x\le\dfrac{\pi}{2}$和$x$軸所圍區域繞$y$軸旋轉之體積
(列出算式4分,計算算式的值6分)
解:



採用圓柱殼法$r=x$, $l(x)=\cos x$

$V=2\pi\displaystyle\int_0^{\frac{\pi}{2}}x\cos x dx=2\pi(x\sin x+\cos x\Big|_0^{\frac{\pi}{0}})=\pi^2-2\pi$